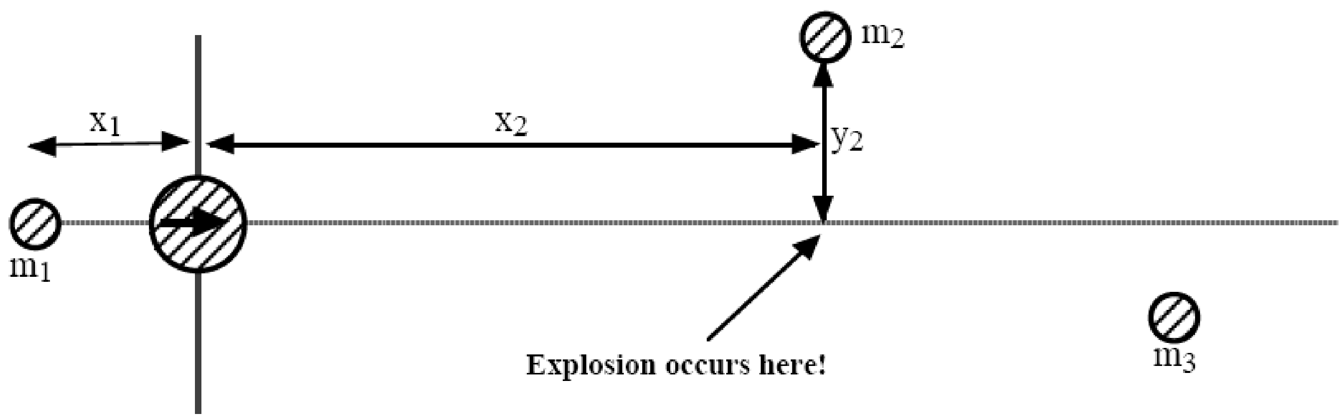
Calculating the Final Locations in an Explosion

On the last test on momentum there seemed to be one [actually, there were two, but we'll get to that at another time] problem that was almost universally missed, the one about the exploding projectile. The key item about this problem is that when the projectile explodes its total momentum remains constant. What this means, from a practical point, of view is that the velocity of the "center of mass" remains constant. This means that when the projectile explodes the velocity of the center of mass continues on as if there had never been an explosion at all.

This means that to solve this problem you need to understand two things: how to calculate the range of a projectile and how to calculate the center of mass of a system of particles. Let's first address calculating the range of a projectile.

To do this I will go through one of the problems from one version of the Momentum Conservation test, 2000-2001.

5. A 22.0 kg projectile is fired with a velocity of 168m/s at an angle of 28.0^{o} above the horizontal. Just as the projectile reaches the highest point of its trajectory this projectile explodes into three pieces m_1 = 4.0 kg, m_2 = 6.0 kg and m_3 = 12.0 kg which are all thrown horizontally. As a result of this explosion m_1 lands x_1 = 220m to the left of the starting point while m_2 lands a distance y_2 = 125m perpendicularly from the exact midpoint of the trajectory. [Note, drawing is NOT to scale!]



a. How long after the explosion will these two pieces reach the ground?
b. What will be the velocity of the center of mass of this system immediately after the projectile explodes?
c. Where will mass m_3 strike the ground?
d. What will be the velocity of each piece of this projectile immediately after the explosion?
e. How much energy was released by this explosion?

As I am sure all of you are aware, to solve a two dimensional projectile problem you need to make 2 data tables; one for the vertical and one for the horizontal. In this case a projectile is fired with a speed of 168m/s at an angle of 28^{o} above the horizontal; break this velocity into vertical and horizontal components.

$$v = 168\frac{m}{s} \quad \alpha = 28deg \quad v_{oh} = v\cdot\cos(\alpha) = 148\frac{m}{s} \quad v_{ov} = v\cdot\sin(\alpha) = 79\frac{m}{s}$$

b. The only velocity at the highest point of the projectile's trajectory will be the horizontal component!

$$V_{highest} = 148m/s$$

Next, make up the 2 tables describing the vertical and horizontal motion of the projectiles.

Vertical	Horizontal
$D_{ov} = 0m$	$D_{oh} = 0m$
$D_{.fv} := "?"$	$D_{fh} = "?"$
$v_{ov} = 79\frac{m}{s}$	$v_{oh} = 148\frac{m}{s}$
$v_{fv} = 0\frac{m}{s}$	$v_{fh} = v_{oh} = 148\frac{m}{s}$
$a = -9.8\frac{m}{s^2}$	$a = 0\frac{m}{s^2}$ $v = 168\frac{m}{s}$
$t = "?"$	$t = "?"$

Using this data table and the vertical data calculate the time to the highest point using $v_f = a \cdot t + v_o$

$$v_{fv} = a + v_{ov} \qquad t = \frac{v_{fv} - v_{ov}}{a} \qquad t = 8.05\,s$$

a. This is the time it will take for the pieces of the projectile to strike the ground ⟶ **8.05s**

Note that this is the time to the highest point. The total time can be calculated by either doubling this time [$t_{total} = 16.1s$] or by using the displacement equation:

$$D_{fv} = 0 = \frac{1}{2} \cdot a \cdot t^2 + v_{ov} \cdot t + D_{ov} = -4.9 \cdot t^2 + 79 \cdot t + 0 \qquad t = \frac{v_{ov}}{-\frac{1}{2} \cdot a} = 16.1\,s$$

Now that you know the total flight time of the projectile you can determine the range of the projectile.

$$D_{fh} = v_{oh} \cdot t = 2388\,m$$

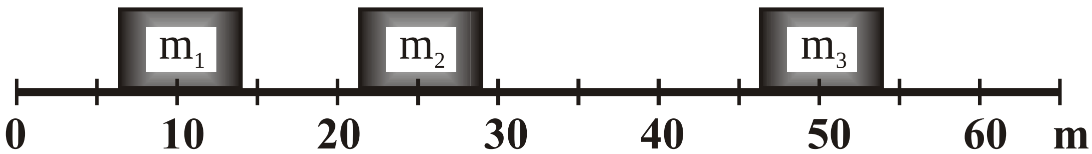
c. The range of the projectiles will be D_{fh} = **2388m**

This range is the same for both the unexploded projectile and for the center of mass since the velocity of the center of mass is not changed by the explosion [momentum is conserved!].

Finding the Center of Mass

The next step will be to use the center of mass calculation to determine the final positions of each object, but first let's look at a sample of how center of mass is determined. As you will find this is, for all practical purposes, exactly what we did in the lab where opposite torques were equal and were used to determine where a single upward force [the center of mass] could lift the system at equilibrium.

Consider the following system of particles consisting of m_1=25kg, m_2 =40kg and m_3=65kg arranged as shown below.



To find the center of mass of this system multiply each mass by its corresponding x coordinate, add each of these products together and then make that sum equal to the total mass of the system multiplied by the $\mathbf{X_{cm}}$ coordinate of the center of mass.

$$\mathbf{m_1 \cdot x_1 + m_2 \cdot x_2 + m_3 \cdot x_3 + m_n \cdot x_n = (m_1 + m_2 + m_3 + m_n) \cdot X_{cm} = M \cdot X_{cm}}$$

In this case this becomes:

$$\mathbf{25 \cdot 10 + 40 \cdot 25 + 65 \cdot 50 = 4500 = (25 + 40 + 65) \cdot X_{cm} = 130 \cdot X_{cm}}$$

Solving for the x coordinate of the center of mass $\mathbf{X_{cm}}$**:**

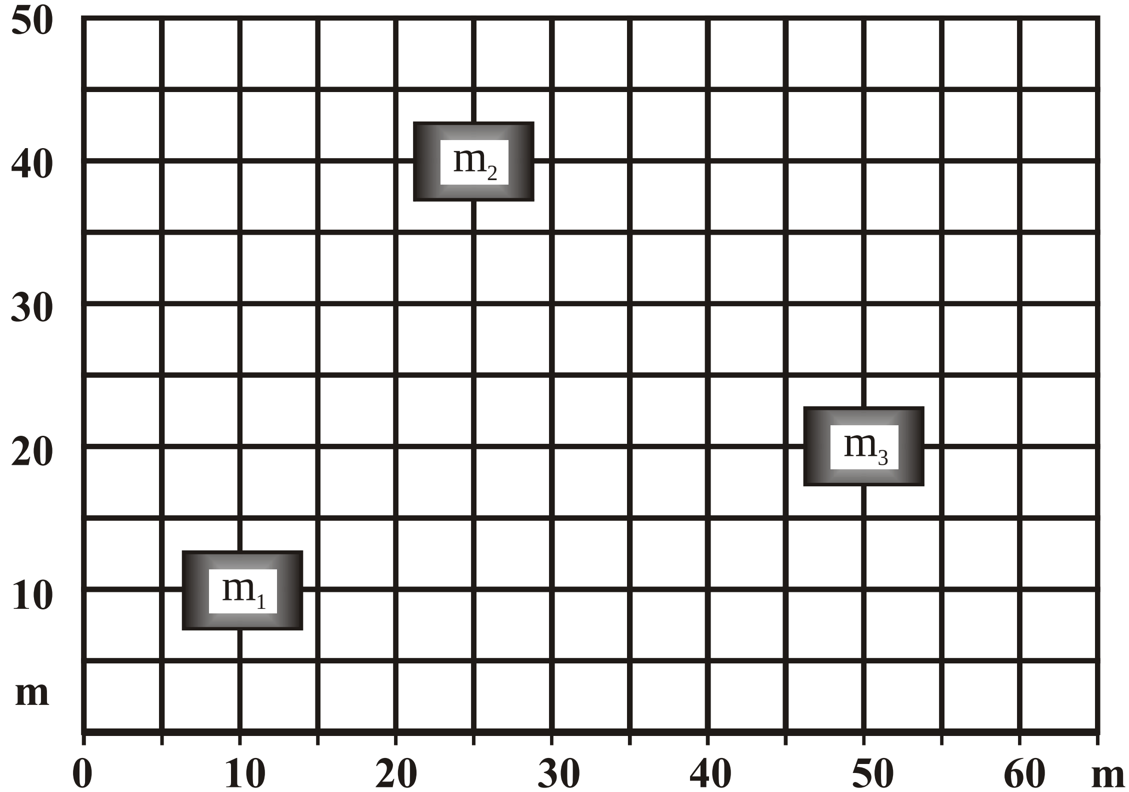
$$\mathbf{X_{cm} = 4500/130 = 34.6m}$$

Notice that everything we did here was exactly what we did in the "Torques at Equilibrium" lab except that there is no "g" that would have been on both sides of the equation in any case and would have cancelled out mathematically!

If the system is 2 dimensional everything is the same except that you will need to repeat this calculation for the y coordinate as well.

$$\mathbf{m_1 \cdot y_1 + m_2 \cdot y_2 + m_3 \cdot y_3 + m_n \cdot y_n = (m_1 + m_2 + m_3 + m_n) \cdot Y_{cm} = M \cdot Y_{cm}}$$

For example, suppose that the same three masses as above are arranged as shown below.



The resulting $\mathbf{X_{cm}}$ coordinate of the center of mass will be the same as above since the x coordinates are unchanged, but now that system has a y coordinate $\mathbf{Y_{cm}}$ of the center of mass as well. To find that $\mathbf{Y_{cm}}$ coordinate you multiply each mass by its corresponding **y** coordinate, add each of these products together and then make this sum equal to the total mass **M** of the system multiplied by the $\mathbf{Y_{cm}}$ coordinate of the center of mass.

$$m_1 \cdot y_1 + m_2 \cdot y_2 + m_3 \cdot y_3 + m_n \cdot y_n = (m_1 + m_2 + m_3 + m_n) \cdot Y_{cm} = M \cdot Y_{cm}$$

In this case this becomes:

$$25 \cdot 10 + 40 \cdot 40 + 65 \cdot 20 = 3150 = (25 + 40 + 65) \cdot Y_{cm} = 130 \cdot Y_{cm}$$

Solving for the y coordinate of the center of mass $\mathbf{Y_{cm}}$**:**

$$Y_{cm} = 3150/130 = 24.2\text{m}$$

If you combine these two results you can determine the two dimensional center of mass of this system of particles:

$$(X_{cm}, Y_{cm}) = (34.6\text{m}, 24.2\text{m})$$

Although I am not going to do it here, the same thing can be done for the $\mathbf{Z_{cm}}$ coordinate, determining the three dimensional center of mass:

$$(X_{cm}, Y_{cm}, Z_{cm})$$

Now that you know how to determine the center of mass of a system of particles, let's use this approach to determine the final position of the particles produced in the problem above.

"A 22.0 kg projectile is fired with a velocity of 168m/s at an angle of 28.0^o above the horizontal. Just as the projectile reaches the highest point of its trajectory this projectile explodes into three pieces m_1 = 4.0 kg, m_2 = 6.0 kg and m_3 = 12.0 kg which are all thrown horizontally. As a result of this explosion m_1 lands x_1 = 220m to the left of the starting point while m_2 lands a distance y_2 = 125m perpendicularly from the exact midpoint of the trajectory."

First remember that the velocity of the center of mass remains constant when the explosion occurs. This means that the center of mass of this system is:

$\mathbf{M \longrightarrow (X_{cm}, Y_{cm}) = (2388m, 0m)}$ [All of this assumes that the origin of this system is the initial launch position of the projectile.]

Now according to the problem, the final positions of the three pieces produced by the explosion are:

$\mathbf{m_1=4.0kg \longrightarrow (-220m, 0m)}$ $\mathbf{m_2=6.0kg \longrightarrow (1194m, 125m)}$ $\mathbf{m_3=12.0kg \longrightarrow (x_3, y_3)}$

With this information we can now find the final coordinates of piece $\mathbf{m_3}$:

$$\mathbf{m_1 \cdot x_1 + m_2 \cdot x_2 + m_3 \cdot x_3 + m_n \cdot x_n = (m_1 + m_2 + m_3 + m_n) \cdot X_{cm} = M \cdot X_{cm}}$$

$$\mathbf{4 \cdot (-220) + 6 \cdot 1194 + 12 \cdot x_3 = (4 + 6 + 12) \cdot 2390 = (22) \cdot 2388 \qquad \text{Solve for } x_3 \longrightarrow x_3 = 3708m}$$

$$\mathbf{m_1 \cdot y_1 + m_2 \cdot y_2 + m_3 \cdot y_3 + m_n \cdot y_n = (m_1 + m_2 + m_3 + m_n) \cdot Y_{cm} = M \cdot Y_{cm}}$$

$$\mathbf{4 \cdot 0 + 6 \cdot 1194 + 12 \cdot y_3 = (4 + 6 + 12) \cdot 0 \quad \text{Solve for } y_3 \longrightarrow x_3 = -597m}$$

Therefore, the final coordinates of the 3rd piece $\mathbf{m_3}$ produced in the explosion of are:

$\mathbf{m_3=12.0kg \longrightarrow (3708m, -597m)}$

Now let's go back and look at the questions asked in the problem.

a. How long after the explosion will these two pieces reach the ground? **[8.05s]**

The time to the highest point will be equal to the time from the highest point back to the ground since the three pieces were each thrown horizontally in the explosion.

b. What will be the velocity of the center of mass of this system immediately after the projectile explodes? **[148m/s]**

The only velocity at the highest point is the horizontal velocity. The vertical component will be zero at the highest point.

c. Where will mass m_3 strike the ground? **[3708m, -597m]**

These are the coordinates of the 3rd piece when it strikes the ground.

d. What will be the velocity of each piece of this projectile immediately after the explosion?

To answer this question divide the final displacement of each of the pieces produced in the explosion divided by the time for each piece to strike the ground.

v_1 = x/t = (-220-1194)/8.05 = -176 m/s i

v_2 = y/t =125/8.05 = 15.5 m/s j

$$v_3 = \frac{\sqrt{x^2+y^2}}{t} = \frac{\sqrt{(3708-1194)^2+-597^2}}{8.05} = 303\,\frac{m}{s} \qquad \alpha = \operatorname{atan}\left[\frac{-597}{(3708-1194)}\right] = -13.4\,deg$$

v_3 =303 m/s at -13.4° [to the right of the original path]

e. How much energy was released by this explosion?

Just calculate the kinetic energy before the explosion, after the explosion and then take the difference!

$$KE_1 = \frac{1}{2}\cdot m_1\cdot v_1^2 = 6.195\times 10^4\,J$$

$$KE_2 = \frac{1}{2}\cdot m_2\cdot v_2^2 = 720.75\,J$$

$$KE_3 = \frac{1}{2}\cdot m_3\cdot v_3^2 = 5.509\times 10^5\,J$$

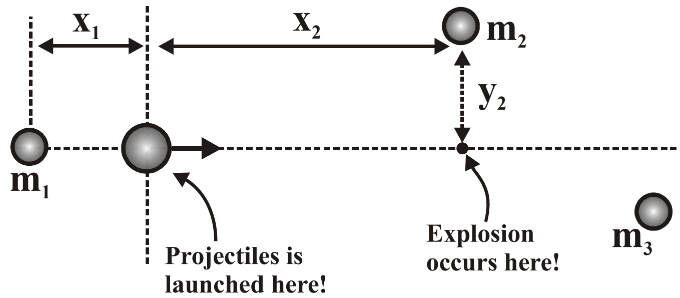
$$KE_f = KE_1 + KE_2 + KE_3 = 6.135\times 10^5\,J$$

$$KE_o = \frac{1}{2}\cdot M\cdot V^2 = 2.409\times 10^5\,J$$

$\Delta KE = KE_f - KE_o$ = 3.726 x 10^5 J are released in the explosion!

I hope all of this has helped because I have put in quite a few hours generating this explanation! I welcome your comments and suggestions.

So now, here is the same question from this year's test! See what you can do!



5. A 21.0 kg projectile is fired with a velocity of 245 m/s at an angle of 28.0^o above the horizontal. Just as the projectile reaches the highest point of its trajectory this projectile explodes into three pieces m_1 = 5.00 kg, m_2 = 7.00 kg and m_3 = 9.00 kg which are all thrown horizontally. As a result of this explosion m_1 lands x_1 = 175 m to the left of the starting point while m_2 lands a distance y_2 = 145 m perpendicularly from the exact midpoint of the trajectory. [Note, drawing is NOT to scale!]
   a. How long after the explosion will these three pieces reach the ground? [5 pts]
   b. What will be the velocity of the center of mass of this system at the time of the explosion? [5 pts]
   c. Where will mass m_3 strike the ground? [5 pts]
   d. What will be the velocity of each piece of this projectile immediately after the explosion? [5 pts]
   e. How much energy was released by this explosion? [5 pts]

More questions to come!