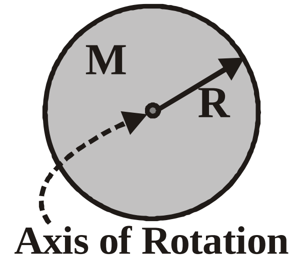
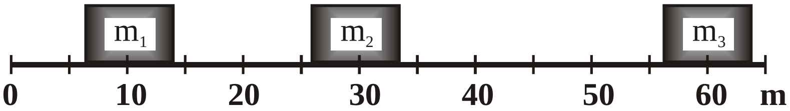
Calculating Moment of Inertia - Integration



Suppose that you would like to determine the Moment of Inertia of a uniform disk, which has a mass of M = 12.0 kg and a radius of R = 15.0cm, about an axis directed through and perpendicular to the center of the disk as shown to the right.

Before we begin this problem, let's first review the discrete problem!

Consider a system consisting of three discrete mass arranged as shown below. Where m_1 = 35kg, m_2 = 54kg and m_3 = 82kg.



The moment of inertia of this system about x = 0m can be determined by using:

$I = m_1 \cdot x_1^2 + m_2 \cdot x_2^2 + m_3 \cdot x_3^2 + m_n \cdot x_n^2$

Which in mathematical notation can be written as:

$$I = \sum_{n=1}^{3} \left(m_n \cdot x_n^2 \right)$$

The moment of inertia I n this case this becomes:

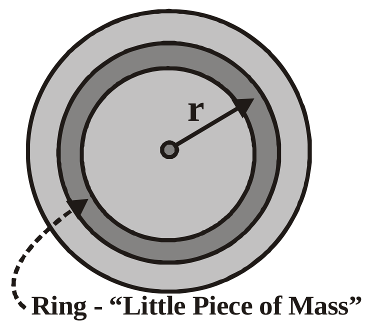
$I_{x=0} = 35 \cdot 10^2 + 54 \cdot 30^2 + 82 \cdot 60^2 = 3.47 \times 10^5\ kg \cdot m^2$ **[Moment of inertia I of the system]**

Now using this same approach let's determine the moment of inertia of the solid disk. But in order to do this we have to first solve a problem. Since the mass of the disk is not all the same distance from the center of rotation we cannot simply multiply the mass of the disk **M** by the radius of the disk **R** squared.

So what is the solution?

What if we broke the disk up into little pieces such that all the mass in each little piece is the same distance from the center of rotation?

If that were true, we could then determine the moment of inertia of each little piece **ΔI** by multiplying the mass of that little piece **Δm** by the distance of that little piece from the center of rotation squared $\mathbf{r^2}$. [$\Delta I = r^2 \cdot \Delta m$]



The question is what would the shape of that little piece have to be so that all the mass in that little piece is approximately the same distance from the center rotation?

The answer is a ring! Notice in the ring at the right that all the mass in this ring is approximately the same distance **r** from the center of the disk, the center of rotation.

So, if we knew the mass of that ring **Δm**, and since all the mass in that ring is approximately the same distance from the center of rotation **r**, we could multiply the mass of that ring **Δm** by the average distance of that mass from the center of rotation squared $\mathbf{r^2}$ to find the moment of inertia of that ring $\Delta I_{ring} = r^2 \cdot \Delta m$.

The next step will be to determine the mass Δm of this ring.

First of all, we can make a pretty good approximation of the mass of this ring by calculating the average mass density of the disk **σ** and then multiply that average mass density by the area **ΔA** of the ring.

The average mass density of the disk **σ** can be found by dividing the total mass **M** of the disk by the total area $\mathbf{A_{disk}}$ of the disk.

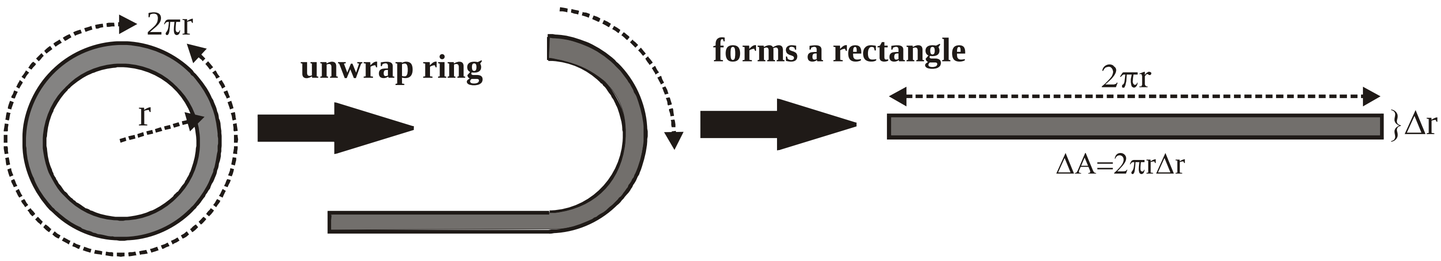
$$\sigma = M/A_{disk}$$

In the case of a disk the area will be $\mathbf{A_{disk} = \pi \cdot R^2}$ and so the mass density per unit area s becomes:

$$\sigma = M/(\pi \cdot R^2)$$

But what is the area ΔA of the ring?

As hard as it may initially seem to believe, we can get a pretty good approximation of the area **ΔA** of this ring by treating the ring as if it were a rectangle. Imagine taking this ring and "unrolling" it into a rectangle as shown below.



[You might argue that this rectangle does not really correspond to the area **ΔA** of the ring, but if the thickness of the ring **Δr** is **VERY** small, then the inner circumference of the ring will be almost the same length as the outer circumference of the ring, in which case it will be quite appropriate to use the area of the rectangle as a very good approximation of the area of the ring. Ultimately, if we allow the thinness of the ring **Δr** to approach zero, the area of the rectangle and the area of the ring will be exactly the same!]

Since the area of a rectangle is equal to the length **L** multiplied by the width **W**, $\mathbf{A_{rectangle} = L \cdot W}$, the area of the ring **ΔA** will be:

$$\Delta A = L \cdot W = 2 \cdot \pi \cdot r \cdot \Delta r$$

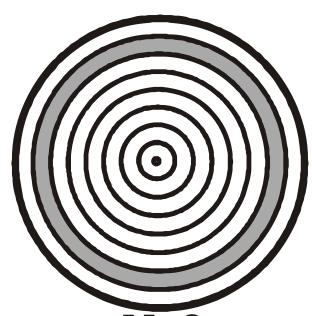
Now that we know the area of the ring **ΔA,** we can now determine the mass of this ring since the mass of the ring will be equal to the product of the mass density of the disk **σ** multiplied by the area **ΔA** of the ring:

$$\Delta m = \sigma \cdot \Delta A = \sigma \cdot [2 \cdot \pi \cdot r \cdot \Delta r] = 2 \cdot \pi \cdot \sigma \cdot r \cdot \Delta r$$

The moment of inertia of this ring **ΔI** can now be determined by multiplying the mass of this ring **Δm** by the distance of this mass from the center of rotation squared $\mathbf{r^2}$.

$$\Delta I = \Delta m \cdot r^2 = 2 \cdot \pi \cdot \sigma \cdot r \cdot \Delta r \cdot r^2 = 2 \cdot \pi \cdot \sigma \cdot r^3 \cdot \Delta r$$

Now that we know the moment of inertia of one ring **ΔI** we can, in principle, determine the moment of inertia $\mathbf{I_{disk}}$ of the entire disk by breaking the disk up into many rings as seen to the right, determine the moment of inertia of each ring and then add these individual moments of inertia together to determine the total moment of inertia of the disk $\mathbf{I_{disk}}$.



N=8

$$I_{disk} = \Delta m_1 \cdot r_1^2 + \Delta m_2 \cdot r_2^2 + \Delta m_3 \cdot r_3^2 + \ldots = \sum_{n=1}^{8} \left(\Delta m_n \cdot r_n^2 \right)$$

If we now take the limit of this sum as the number of rings goes to infinity, we will get rid of any error introduced by our approximation of the area of the ring as being the same as the area of a rectangle and we will, likewise, eliminate any error caused by some mass in the ring being closer or farther from the center of rotation.

$$I_{disk} = \lim_{N \to \infty} \sum_{n=1}^{N} \left(\Delta m_n \cdot r_n^2 \right)$$

Taking the limit of this sum is, of course, the same thing as taking the integral of $\mathbf{r^2 \cdot \Delta m}$!

$$I_{disk} = \lim_{N \to \infty} \sum_{n=1}^{N} \left(\Delta m_n \cdot r_n^2 \right) = \int_0^R r^2 \, dm$$

Remember from above that the mass of each individual ring is given by:

$$\mathbf{\Delta m = \sigma \cdot \Delta A = \sigma \cdot [2 \cdot \pi \cdot r \cdot \Delta r] = 2 \cdot \pi \cdot \sigma \cdot r \cdot \Delta r}$$

As we allow the number of rings to approach infinity the differential $\mathbf{\Delta m}$ becomes $\mathbf{dm}$, $\mathbf{\Delta A}$ becomes $\mathbf{dA}$, while the differential $\mathbf{\Delta r}$ becomes $\mathbf{dr}$.

$$\mathbf{dm = \sigma \cdot dA = \sigma \cdot [2 \cdot \pi \cdot r \cdot dr] = 2 \cdot \pi \cdot \sigma \cdot r \cdot dr}$$

Therefore, the moment of inertia of the disk $\mathbf{I_{disk}}$ becomes:

$$I_{disk} = \int_0^R r^2 \, dm = \int_0^R r^2 \cdot (2 \cdot \pi \cdot \sigma \cdot r) \, dr = 2 \cdot \pi \cdot \sigma \cdot \int_0^R r^3 \, dr$$

$$I_{disk} = 2 \cdot \pi \cdot \sigma \cdot \int_0^R r^3 \, dr = 2 \cdot \pi \cdot \sigma \cdot \frac{r^4}{4} = \frac{1}{2} \cdot \pi \, \sigma \cdot R^4 - \frac{1}{2} \cdot \pi \cdot \sigma \cdot 0^4 = \frac{1}{2} \cdot \pi \, \sigma \cdot R^4$$

But the total mass **M** of the disk is equal to the mass density $\boldsymbol{\sigma}$ of the disk multiplied by the total area $\mathbf{A = \pi \cdot R^2}$ of the disk.

$$\mathbf{M = \sigma \cdot A = \sigma \cdot (\pi \cdot R^2) = \pi \cdot \sigma \cdot R^2}$$

Therefore, the moment of inertia of the disk will be:

$$\mathbf{I_{disk} = ½ \cdot \pi \cdot \sigma \cdot R^4 = ½ \cdot [\pi \cdot \sigma \cdot R^2] \cdot R^2 = ½ \cdot M \cdot R^2}$$

Remembering from the beginning of this discussion that our disk has a mass of M = 12·kg and a radius of R = 0.15·m, the moment of inertia of this uniform solid disk $\mathbf{I_{disk}}$ becomes:

$$\mathbf{I_{disk} = ½ \cdot 12 \cdot 0.15^2 = 0.135 kg \cdot m^2}$$